

FP1 JAN 13

1. Show, using the formulae for $\sum_{r=1}^n r$ and $\sum_{r=1}^n r^2$, that

$$\sum_{r=1}^n 3(2r-1)^2 = n(2n+1)(2n-1), \text{ for all positive integers } n.$$

$$\sum 12r^2 - 12r + 3 = 12 \sum r^2 - 12 \sum r + 3 \sum 1$$

$$= 12 \times \frac{1}{6} n(n+1)(2n+1) - 12 \times \frac{1}{2} n(n+1) + 3n$$

$$= n [2(n+1)(2n+1) - 6(n+1) + 3]$$

$$= n [4n^2 + \cancel{6n} + 2 - \cancel{6n} - 6 + 3]$$

$$= n [4n^2 - 1] = n(2n+1)(2n-1) \quad \#$$

2.

$$z = \frac{50}{3+4i}$$

Find, in the form $a+ib$ where $a, b \in \mathbb{R}$,

(a) z ,

(b) z^2 .

(2)

Find

(c) $|z|$,

(2)

(d) $\arg z^2$, giving your answer in degrees to 1 decimal place.

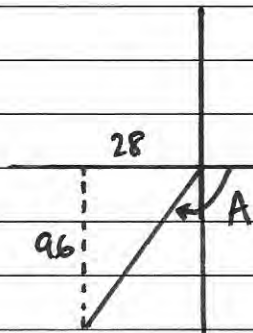
(2)

$$a) \frac{50(3-4i)}{(3+4i)(3-4i)} = \frac{150-200i}{9+16} = \frac{150-200i}{25} = 6-8i$$

$$b) z^2 = (6-8i)(6-8i) = 36-64-96i = -28-96i$$

$$c) |z| = \sqrt{6^2+8^2} = 10$$

$$d) \arg z^2$$



$$A = 180 - \tan^{-1}\left(\frac{96}{28}\right)$$

$$A = 106.3$$

$$\therefore \arg z^2 = -106.3$$

3.

$$f(x) = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 5, \quad x > 0$$

(a) Find $f'(x)$.The equation $f(x) = 0$ has a root α in the interval $[4.5, 5.5]$.(b) Using $x_0 = 5$ as a first approximation to α , apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α , giving your answer to 3 significant figures.

(4)

$$a) f'(x) = x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$b) x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{2(5)^{\frac{1}{2}} + (5)^{-\frac{1}{2}} - 5}{(5)^{-\frac{1}{2}} - \frac{1}{2}(5)^{-\frac{3}{2}}}$$

$$x_1 = 5.200377653.. \quad \therefore x_1 = \underline{5.20} \text{ (3sf)}$$

4. The transformation U , represented by the 2×2 matrix P , is a rotation anticlockwise about the origin.

(a) Write down the matrix P .

The transformation V , represented by the 2×2 matrix Q , is a reflection in the line $y = -x$.

(b) Write down the matrix Q .

(1)

Given that U followed by V is transformation T , which is represented by the matrix R ,

(c) express R in terms of P and Q ,

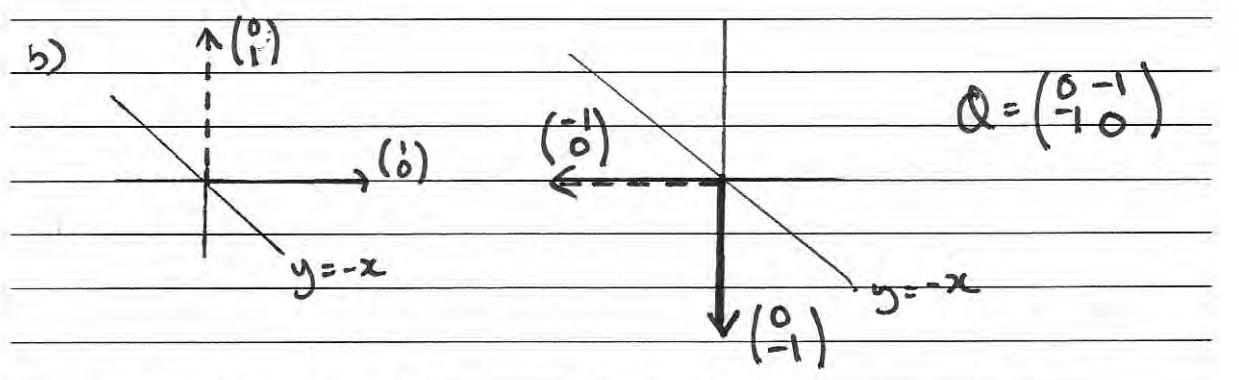
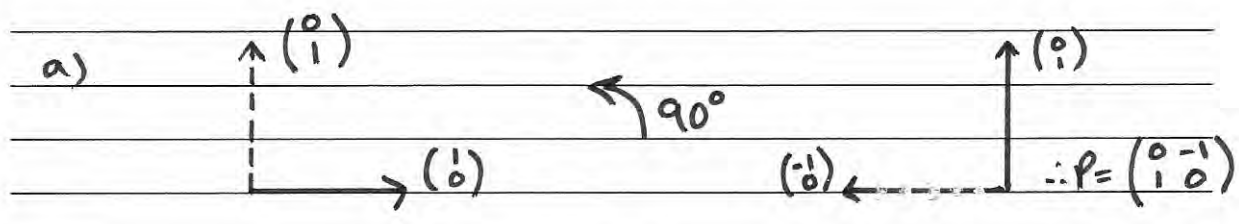
(1)

(d) find the matrix R ,

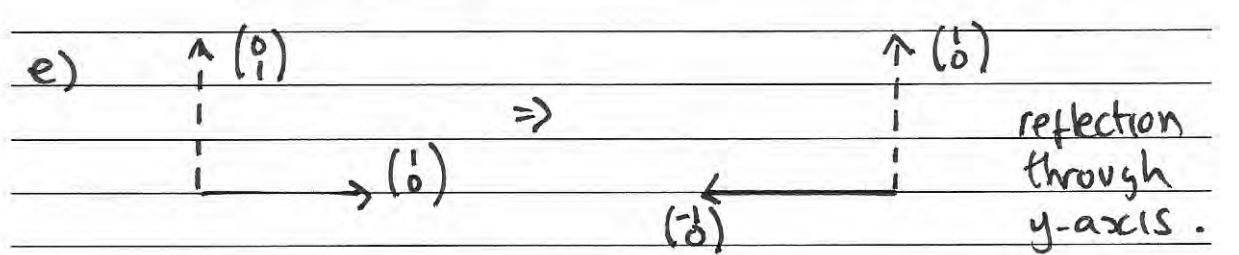
(2)

(e) give a full geometrical description of T as a single transformation.

(2)



d) $R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = (d)$ $R = VU$
 c) $\therefore R = QP$



$$f(x) = (4x^2 + 9)(x^2 - 6x + 34)$$

(a) Find the four roots of $f(x) = 0$

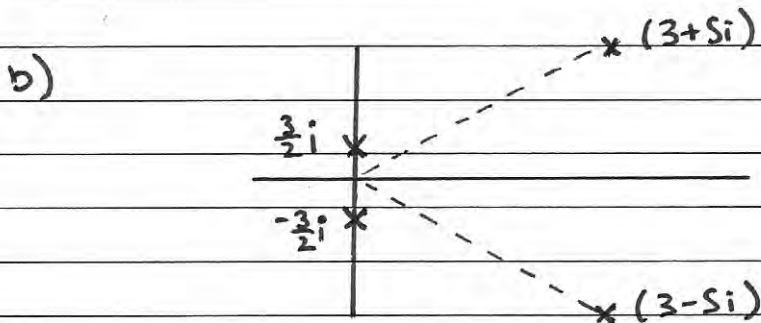
Give your answers in the form $x = p + iq$, where p and q are real.

(b) Show these four roots on a single Argand diagram.

(2)

$$4x^2 + 9 = 0 \Rightarrow x^2 = -\frac{9}{4} \Rightarrow x = \pm \sqrt{-\frac{9}{4}} \quad x = \frac{3}{2}i, -\frac{3}{2}i$$

$$\begin{aligned} x^2 - 6x + 34 = 0 &\Rightarrow (x-3)^2 - 9 = -34 \\ &\Rightarrow (x-3)^2 = -25 \\ &\Rightarrow x-3 = \pm 5i \Rightarrow x = 3+5i, 3-5i \end{aligned}$$



6.

$$\mathbf{X} = \begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix}, \text{ where } a \text{ is a constant.}$$

(a) Find the value of a for which the matrix \mathbf{X} is singular.

$$\mathbf{Y} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

(b) Find \mathbf{Y}^{-1} .

(2)

The transformation represented by \mathbf{Y} maps the point A onto the point B .Given that B has coordinates $(1 - \lambda, 7\lambda - 2)$, where λ is a constant,(c) find, in terms of λ , the coordinates of point A .

(4)

$$a) \det \mathbf{X} = 2 - 3a = 0 \quad \therefore a = \frac{2}{3}$$

$$b) \det \mathbf{Y} = 2 - (-3) = 5$$

$$\therefore \mathbf{Y}^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}$$

$$c) \mathbf{Y}\mathbf{A} = \mathbf{B} \Rightarrow \mathbf{Y}^{-1}\mathbf{Y}\mathbf{A} = \mathbf{Y}^{-1}\mathbf{B} \Rightarrow \mathbf{A} = \mathbf{Y}^{-1}\mathbf{B}$$

$$\therefore \mathbf{A} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 - \lambda \\ 7\lambda - 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 - 2\lambda + 7\lambda - 2 \\ -3 + 3\lambda + 7\lambda - 2 \end{pmatrix}$$

$$\therefore \mathbf{A} = \frac{1}{5} \begin{pmatrix} 5\lambda \\ 10\lambda - 5 \end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix} \lambda \\ 2\lambda - 1 \end{pmatrix} \quad \mathbf{A}(\lambda, 2\lambda - 1)$$

7. The rectangular hyperbola, H , has cartesian equation $xy = 25$

The point $P \left(5p, \frac{5}{p} \right)$, and the point $Q \left(5q, \frac{5}{q} \right)$, where $p, q \neq 0, p \neq q$, are on the rectangular hyperbola H .

(a) Show that the equation of the tangent at point P is

$$p^2 y + x = 10p \quad (4)$$

(b) Write down the equation of the tangent at point Q . (1)

The tangents at P and Q meet at the point N .

Given $p + q \neq 0$,

(c) show that point N has coordinates $\left(\frac{10pq}{p+q}, \frac{10}{p+q} \right)$. (4)

The line joining N to the origin is perpendicular to the line PQ .

(d) Find the value of $p^2 q^2$. (5)

$$a) y = 25x^{-1} \Rightarrow \frac{dy}{dx} = -25x^{-2} = \frac{-25}{x^2} \quad x = 5p \therefore \frac{dy}{dx} = \frac{-1}{p^2}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - \frac{5}{p} = \frac{-1}{p^2}(x - 5p)$$

$$\Rightarrow p^2 y - 5p = -x + 5p \Rightarrow p^2 y + x = 10p \quad \#$$

$$b) q^2 y + x = 10q$$

$$c) x = 10p - p^2 y = 10q - q^2 y$$

$$\Rightarrow 10(p - q) = (p^2 - q^2)y$$

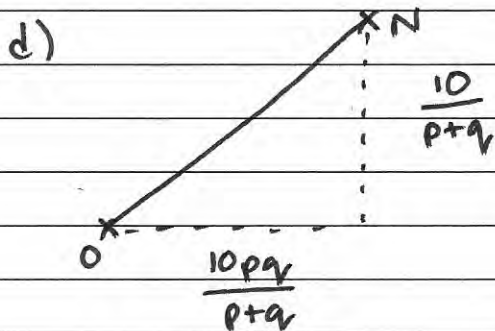
$$\Rightarrow 10(p - q) = (p + q)(p - q)y \quad \therefore y = \frac{10}{p + q}$$

$$x = 10p - p^2y = 10p - p^2 \left(\frac{10}{p+q} \right)$$

$$x = \frac{10p(p+q) - 10p^2}{p+q} = \frac{10p^2 + 10pq - 10p^2}{p+q}$$

$$\therefore x = \frac{10pq}{p+q}$$

$$\therefore N \left(\frac{10pq}{p+q}, \frac{10}{p+q} \right)$$



$$\therefore M_{ON} = \frac{\frac{10}{p+q}}{\frac{10pq}{p+q}} = \frac{10}{10pq} = \frac{1}{pq}$$

$$\therefore M_{PO} = -pq$$

$$P \left(s_p, \frac{s}{p} \right) \quad Q \left(s_q, \frac{s}{q} \right)$$

$$M_{PQ} = \frac{\frac{s}{q} - \frac{s}{p}}{s_q - s_p} = \frac{s_p - s_q}{pq} \cdot \frac{1}{-(s_p - s_q)}$$

$$\Rightarrow M_{PO} = \frac{-1}{pq} \quad \therefore -pq = \frac{-1}{pq} \Rightarrow -p^2q^2 = -1$$

$$\therefore \underline{p^2q^2 = 1}$$

8. (a) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^n r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

(b) A sequence of positive integers is defined by

$$u_1 = 1, \\ u_{n+1} = u_n + n(3n+1), \quad n \in \mathbb{Z}^+$$

Prove by induction that

$$u_n = n^2(n-1) + 1, \quad n \in \mathbb{Z}^+ \quad (5)$$

$$n=1 \quad \sum_{r=1}^1 r(r+3) = 1(4) = 4 \quad \frac{1}{3}n(n+1)(n+5) = \frac{1}{3}(1)(2)(6) = 4$$

\therefore true for $n=1$

$$n=2 \quad \sum_{r=1}^2 r(r+3) = 2(5) + 4 = 14 \quad \frac{1}{3}n(n+1)(n+5) = \frac{1}{3}(2)(3)(7) = 14$$

\therefore true for $n=2$

$$\text{assume true for } n=k \quad \therefore \sum_{r=1}^k r(r+3) = \frac{1}{3}k(k+1)(k+5)$$

$$n=k+1 \quad k+1: \\ \sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+2)(k+6)$$

$$\sum_{r=1}^{k+1} r(r+3) = \sum_{r=1}^k r(r+3) + (k+1)(k+4)$$

$$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4) = \frac{1}{3}(k+1)[k(k+5) + 3(k+4)]$$

$$= \frac{1}{3}(k+1)[k^2 + 8k + 12] = \frac{1}{3}(k+1)(k+2)(k+6) \quad \#$$

true for $n=1, n=2$. If true for $n=k$, then true for $n=k+1$.
Therefore, by induction true for all $n \in \mathbb{Z}^+$

$$b) \quad u_1 = 1$$

$$(n=1) \quad u_2 = 1 + 1(4) = 5$$

$$(n=2) \quad u_3 = 5 + 2(7) = 19$$

$$u_1 = 1^2(1-1) + 1 = 1$$

$$u_2 = 2^2(2-1) + 1 = 5 \Rightarrow \text{true for } n=1$$

$$u_3 = 3^2(3-1) + 1 = 19 \Rightarrow \text{true for } n=2$$

assume true for $n=k$ so $u_{k+1} = u_k + u(3k+1)$

$$u_k = k^2(k-1) + 1 \Rightarrow u_{k+1} = (k+1)^2(k+1-1) + 1 = u(k+1)^2 + 1$$

$$n=k+1$$

$$u_{k+1} = u_k + u(3k+1)$$

$$= (k^2(k-1) + 1) + u(3k+1)$$

$$= k^3 - k^2 + 1 + 3k^2 + k$$

$$= k^3 + 2k^2 + k + 1$$

$$= k(k^2 + 2k + 1) + 1$$

$$= k(k+1)^2 + 1 \quad \#$$

\therefore true for $n=1, 2$

if true for $n=k$, then
true for $n=k+1$

\therefore by induction, true
for all $n \in \mathbb{Z}^+$

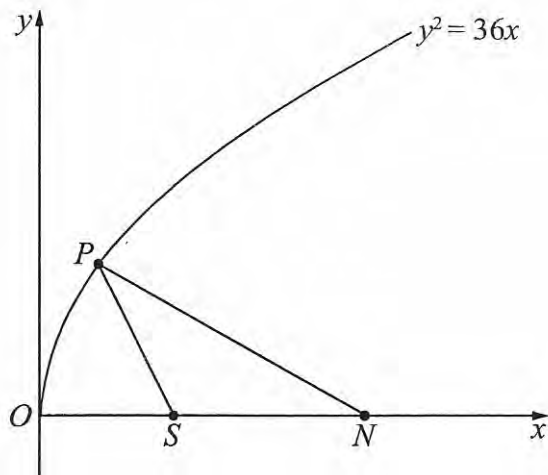


Figure 1

Figure 1 shows a sketch of part of the parabola with equation $y^2 = 36x$.

The point $P(4, 12)$ lies on the parabola.

(a) Find an equation for the normal to the parabola at P .

(5)

This normal meets the x -axis at the point N and S is the focus of the parabola, as shown in Figure 1.

(b) Find the area of triangle PSN .

(4)

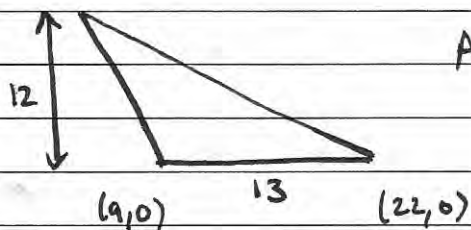
$$y^2 = 36x \quad y^2 = 4ax \quad \therefore a = 9 \quad S(9, 0)$$

$$\frac{d}{dx} y^2 = \frac{d}{dx} 36x \Rightarrow 2y \frac{dy}{dx} = 36 \quad \therefore \frac{dy}{dx} = \frac{18}{y}$$

$$y = 12 \Rightarrow M_t = \frac{18}{12} = \frac{3}{2} \Rightarrow M_n = -\frac{2}{3}$$

$$y - 12 = -\frac{2}{3}(x - 4)$$

$$b) \quad y = 0 \Rightarrow -36 = -2x + 8 \Rightarrow 2x = 44 \Rightarrow x = 22$$



$$\text{Area} = \frac{1}{2}bh = \frac{1}{2} \times 13 \times 12 = \underline{78}$$